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Pattern formation for a type of reaction diffusion system with cross diffusion

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ABSTRACT

In this paper, pattern formation for a Schnakenberg model is studied in one and two dimensions. The model has been studied when the diffusion is nonlinear and so called cross diffusion. The conditions of diffusion driven instability are applied to this model and shown that this model can formulate patterns, and the existence of bifurcation for specific parameters are shown and for different values of wave number k. The use of COMSOL Multiphysics finite element package in simulation shows nice graphs of pattern formations in two dimensions.

1. Introduction

The topic that attracts a large number of researchers especially mathematicians is the pattern formation or Turing models. This connects two different sciences to formulate models for interesting topics in mathematical biology through the study of how structures and patterns in nature develop over time. The mechanism of two chemicals as a pattern formation is studied first using the reaction diffusion equations time by the scientist Turing (1952). A self-organized pattering when diffusion linear is derived for this model by Gambino et al. (2015). However, this isn't the only phenomenon that is formulated when there is an interaction between reaction and diffusion, but many others are formulated in geology, geography, chemistry, industrial process, networks of electrical circuits and of course, mathematics. There are two cases of diffusion that are used in the reaction diffusion system to produce the pattern; the case of each species depends on the gradient of concentration itself which is called self-diffusion. The second case is when the gradient of the density of one species induces a flux of another species, and this is a cross diffusion. Both self- and cross-diffusion terms are common in the context of population dynamics and today appear in different topics like chemotaxis, ecology, social systems, turbulent transport in plasmas, drift- diffusion in semiconductors, granular materials and cell division in tumor growth. The drawback point that can be seen in similar models in population ecology is that most of the papers are focused on mathematical properties in the reaction diffusion system and neglect the idea of pattern formation (Gambino et al., 2012). There is a model in which the pattern formation cannot be seen in it shown in previous studies. This model is called Lotka-Volterra competition-diffusion system with constant diffusion coefficients. However, both of Shigesada et al. (1979) showed that when diffusion terms are nonlinear and the population densities are u and v for two competing species, then the formula of pattern formation can be constructed. The work of Gambino et al. (2015) was introduced the following reaction-diffusion system (Eq. 1):

$$\frac{\partial u}{\partial t} = \nabla^2 u + d_v \nabla^2 v + \gamma f(u,v),$$
$$\frac{\partial v}{\partial t} = d\nabla^2 u + d_u \nabla^2 u + \gamma g(u,v) \quad (1)$$

where, ∇^2 is the bi dimensional Laplacian operator, d is the ratio of the linear diffusion coefficients, d_u and d_v are respectively the ratios of the cross-diffusion and the diffusion coefficients, and γ is a positive constant. The nonlinear kinetics (Eq. 2) describes the Schnakenberg chemical reaction:

$$f(u,v) = a - u + u^2 v,$$
$$\mathrm{g(u,v)} = b - u^2 v. \quad (2)$$

Also, it is required that (1) and (2) be equipped with the following initial conditions:

$$u(x,y,0) = u_0(x,y), v(x,y,0) = v_0(x,y), \ (x,y) \in [0,l_x] \times [0,l_y]$$

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where, l_x and l_y are characteristic lengths. The reaction diffusion model with nonlinear cross-diffusion system which describes segregation effects for competing species in population ecology is known as Shigesada et al. (1979) cross-diffusion system. The models shown in Madzvamuse et al. (2015) and Rasheed (2014) stated that cross-diffusion is responsible for Turing instability. Even when cross diffusion coefficients are linear or small or negative as studied in Vanag and Epstein (2009), it is sufficient to formulate pattern formation.

Moreover, in the above mentioned papers, the diffusion is coupled with nonlinear kinetic terms. Schnakenberg model has been studied with linear cross diffusion to produce the pattern formation in Madzvamuse et al. (2015). When the cross-diffusion terms are absent, the need of constant diffusion coefficient for inhibitor to be large is necessary, whilst the existence of cross diffusion coefficient provides the diffusion constant d so that it is not greater than one. The proposed finite volume method by Andreianov et al. (2011) is used to study the reaction diffusion system with cross diffusion numerically. This model represents a two-species Lotka–Volterra reaction–diffusion competition plank tonic system, and it was shown that the cross diffusion driven instability and patterns will formulated. For more details about using numerical methods to solve Reaction diffusion system see Barrett et al. (2004), Barrio et al. (1999), and Tory et al. (2011).

2. Schnakenberg model with cross diffusion

In this section, we start discussing the conditions of diffusion driven instability in homogeneous cases. The dynamics of pattern formation occurs when the stability of steady state changes after we add cross diffusion. We consider the two species reaction-diffusion system (Eq. 3):

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u,v),\ \frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + g(u,v), \tag{3}$$

with fluxes as in the following expression (Eq. 4):

$$D_u = \nabla[u(c_1 + a_1 u + b_1 v)],$$
$$D_v = \nabla[v(c_2 + a_2 v + b_2 u)]. \tag{4}$$

The reaction terms are (Eq. 5):

$$f(u,v) = \alpha - uv^2,$$
$$g(u,v) = \beta + uv^2 - v. \tag{5}$$

In (3) $u(x,t)$ and $v(x,t)$ with $x \in \Omega$, $\Omega \subseteq R^n$ are the population densities of two interaction species. The parameters $a_i \geq 0$ and $c_i \geq 0$ are respectively the self-diffusion and the diffusion coefficients, while the parameters b_1 and b_2, the cross-diffusion coefficients, are both nonnegative therefore both species are in a competitive relationship (Gambino et al., 2007). Assume that the reaction term has a non-zero homogeneous steady state(u_0, v_0). This system exhibits the diffusion driven instability if the homogeneous steady state (u_0, v_0) is stable to spatially homogeneous perturbations, but unstable to some non-homogeneous perturbations (Rasheed, 2014).

3. Linear stability analysis

Turing reaction-diffusion models are generally non-linear. As such, it can be difficult to understand how a particular solution will develop over time and space. We can gain some understanding of the behavior of the solution by looking at one solution over time. We first linearize the reaction function about the homogeneous steady state solution. Then, the linear stability analysis looks at the time component of a particular solution to see what growth rates will converge to zero, producing a stable-state. From this, we can look at the conditions for which instabilities can occur. For simplicity, we carry out the analysis in one spatial dimension.

The uniform stationary state solution,

$$(u_0, v_0) = \left(\frac{\alpha}{(\alpha+\beta)^2}, \alpha + \beta\right)$$

satisfies

$$f(u_0, v_0) = g(u_0, v_0) = 0.$$

The linearized system in the neighborhood of (u_0, v_0) is (Eqs. 6 and 7):

$$\dot{w} = Kw + D\nabla^2 w,\ \ w = \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix} \tag{6}$$
$$K = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix},$$

or

$$K = \begin{bmatrix} -v_0^2 & -2u_0 v_0 \\ v_0^2 & 2u_0 v_0 - 1 \end{bmatrix},$$
$$D = \begin{bmatrix} c_1 + 2a_1 u_0 + b v_0 & b u_0 \\ b_2 v_0 & c_2 + 2a_2 v_0 + b_2 u_0 \end{bmatrix}. \tag{7}$$

First, we show that the system (3) is stable without reaction terms and satisfies the two conditions:

$$\text{I: Trace(K)} = f_u + g_v < 0,$$
$$\text{Trace (K)} = -v_0^2 + 2u_0 v_0 - 1 < 0,$$
$$-(\alpha+\beta)^2 + 2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta) - 1 < 0,$$

$$\text{II: Det}(K) = f_u g_v - f_v g_u > 0,$$
$$\text{Det }(K) = -v_0^2(2u_0 v_0 - 1) - (-2u_0 v_0)v_0^2 > 0,$$
$$-(\alpha+\beta)^2\left(2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta) - 1\right) - \left(-2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta)\right)(\alpha+\beta)^2 > 0.$$

Next, the stability of steady state changes after we add diffusion. Looking for solutions of system (6) of the form $W_k = C_k e^{\lambda_k t} e^{ikx}$ leads to the following dispersion relation, which gives the eigenvalue λ as a function of the wave number K: $\lambda I = -k^2 D + K$, where, I is an identity matrix,

$$\begin{bmatrix} \lambda - I & 0 \\ 0 & \lambda - I \end{bmatrix} = -k^2 \begin{bmatrix} c_1 + 2a_1u_0 + bv_0 & bu_0 \\ b_2v_0 & c_2 + 2a_2v_0 + b_2u_0 \end{bmatrix} + \begin{bmatrix} -v_0^2 & -2u_0v_0 \\ v_0^2 & 2u_0v_0 - 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda + k^2(c_1 + 2a_1u_0 + bv_0) + v_0^2 & k^2bu_0 + 2u_0v_0 \\ k^2b_2v_0 - v_0^2 & \lambda + k^2(c_2 + 2a_2v_0 + b_2u_0) - (2u_0v_0 - 1) \end{bmatrix} = 0, \text{ or}$$

$$(\lambda + k^2(c_1 + 2a_1u_0 + bv_0) + v_0^2)(\lambda + k^2(c_2 + 2a_2v_0 + b_2u_0) - (2u_0v_0 - 1)) - (k^2b_2v_0 - v_0^2)(k^2bu_0 + 2u_0v_0) = 0$$

$$\lambda^2 + k^2(c_2 + 2a_2v_0 + b_2u_0)\lambda - (2u_0v_0 - 1)\lambda + k^2(c_1 + 2a_1u_0 + bv_0)\lambda + k^4(c_1 + 2a_1u_0 + bv_0)(c_2 + 2a_2v_0 + b_2u_0) - k^2(c_1 + 2a_1u_0 + bv_0)(2u_0v_0 - 1) + v_0^2\lambda + k^2(c_2 + 2a_2v_0 + b_2u_0)v_0^2 - (2u_0v_0 - 1)v_0^2 - (b_2v_0bu_0k^4 + 2b_2u_0v_0^2k^2 - bv_0^2u_0k^2 - 2u_0v_0^3) = 0$$

$$\lambda^2 + \left(k^2\left((c_2 + 2a_2v_0 + b_2u_0) + (c_1 + 2a_1u_0 + bv_0)\right) - \left((2u_0v_0 - 1) - v_0^2\right)\right)\lambda + k^4\left((c_1 + 2a_1u_0 + bv_0)(c_2 + 2a_2v_0 + b_2u_0) - b_2v_0bu_0\right) + \left((c_2 + 2a_2v_0 + b_2u_0)v_0^2 - (c_1 + 2a_1u_0 + bv_0)(2u_0v_0 - 1) + 2b_2u_0v_0^2 - bv_0^2u_0\right)k^2 + (-(2u_0v_0 - 1)v_0^2 - 2u_0v_0^3) = 0$$

$$\lambda^2 + \left(k^2\left(\left(c_2 + 2a_2(\alpha+\beta) + b_2\frac{\alpha}{(\alpha+\beta)^2}\right) + \left(c_1 + 2a_1\frac{\alpha}{(\alpha+\beta)^2} + b(\alpha+\beta)\right)\right) - \left(\left(2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta) - 1\right) - (\alpha+\beta)^2\right)\right)\lambda$$
$$+k^4\left(\left(c_1 + 2a_1\frac{\alpha}{(\alpha+\beta)^2} + b(\alpha+\beta)\right)\left(c_2 + 2a_2(\alpha+\beta) + b_2\frac{\alpha}{(\alpha+\beta)^2}\right) - b_2(\alpha+\beta)b\frac{\alpha}{(\alpha+\beta)^2}\right) + \left(\left(c_2 + 2a_2(\alpha+\beta) + b_2\frac{\alpha}{(\alpha+\beta)^2}\right)(\alpha+\beta)^2 - \left(c_1 + 2a_1\frac{\alpha}{(\alpha+\beta)^2} + b(\alpha+\beta)\right)\left(2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta) - 1\right) + 2b_2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta)^2 - b(\alpha+\beta)^2\frac{\alpha}{(\alpha+\beta)^2}\right)k^2 +$$
$$\left(-\left(2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta) - 1\right)(\alpha+\beta)^2 - 2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta)^3\right) = \lambda^2 + \left(k^2 trace(D) - trace(k)\right)\lambda + h(k^2) = 0 \quad (8)$$

where,

$$h(k^2) = det(D)k^4 + qk^2 + det(k) = 0 \quad (9)$$

with

$$q = (c_2 + 2a_2v_0 + b_2u_0)v_0^2 - (c_1 + 2a_1u_0 + bv_0)(2u_0v_0 - 1) + 2b_2u_0v_0^2 - bv_0^2u_0 \quad (10)$$
$$q = \left(c_2 + 2a_2(\alpha+\beta) + b_2\frac{\alpha}{(\alpha+\beta)^2}\right)(\alpha+\beta)^2 - \left(c_1 + 2a_1\frac{\alpha}{(\alpha+\beta)^2} + b(\alpha+\beta)\right)\left(2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta) - 1\right) + 2b_2\frac{\alpha}{(\alpha+\beta)^2}(\alpha+\beta)^2 - b(\alpha+\beta)^2\frac{\alpha}{(\alpha+\beta)^2}.$$

In order to have $Re(\lambda) > 0$, for some $k \neq 0$,we need trace(D)>0, trace(K)>0 and $h(k^2) < 0$. This implies that, for Turing instability, the following two conditions must hold:

$$q < 0, min\left(h(k^2)\right) < 0 \Leftrightarrow q^2 - 4det(D)det(K) > 0.$$

4. Results

Fig. 1 shows that $h(k^2)$ decreased to be negative when $\alpha > 0.4$ and this will guarantee the existence of Turing instability or pattern formation. When $\alpha = 0.4$, the bifurcation occurs and this parameter will called a bifurcation point.

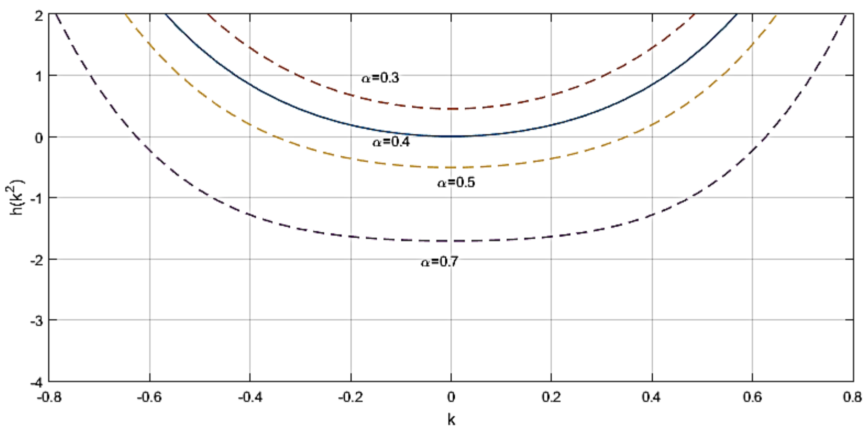


Fig. 1: Comparison between k and $h(k^2)$ for different values of parameter α

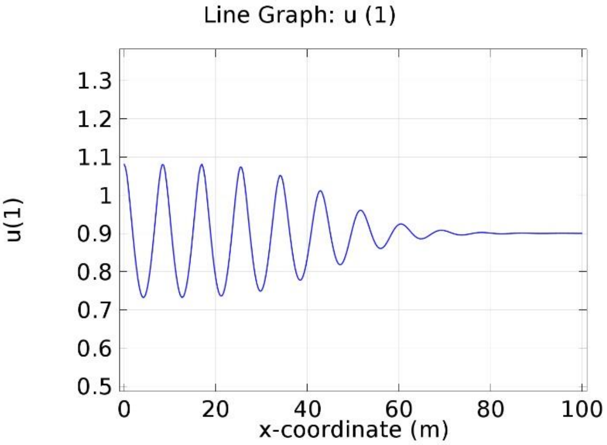


Fig. 2: The numerical solution using COMSOL to show the pattern formation dynamics in u, when $t = 10$ is a type step and the initial condition that we use, $u_0(x)$ and $v_0(x)$ are step functions e^{-x^2} and the parameters are $a_1 = 0.0004$, $a_2 = 0.1$, $c_i = 0.2$,$b_1 = 6.5$, $b_2 = 0.3$, $\alpha = 0.9$ and $\beta = 0.1$

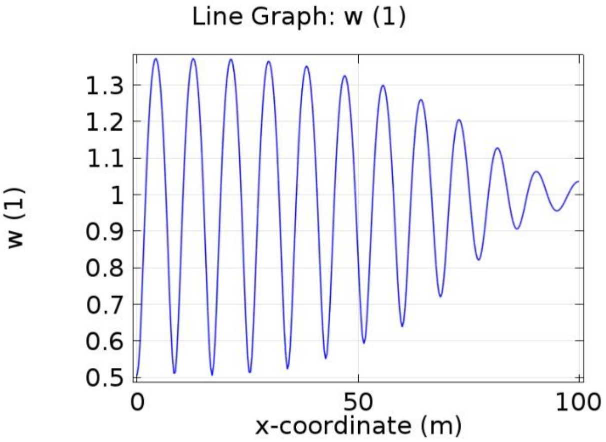
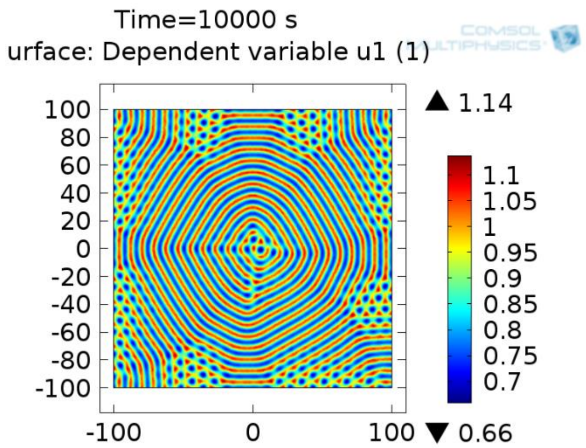


Fig. 3: The numerical solution using COMSOL for Shnakenberg model in (1) which shows the pattern formation dynamics in v, when $t = 10$ is a time step and the initial conditions that we use, $u_0(x)$ and $v_0(x)$ are step functions of the form e^{-x^2} and the parameters are $a_1 = 0.0004$, $a_2 = 0.1$, $c_i = 0.2$, $b = 6.5$, $b_2 = 0.3$, $\alpha = 0.9$, and $\beta = 0.1$



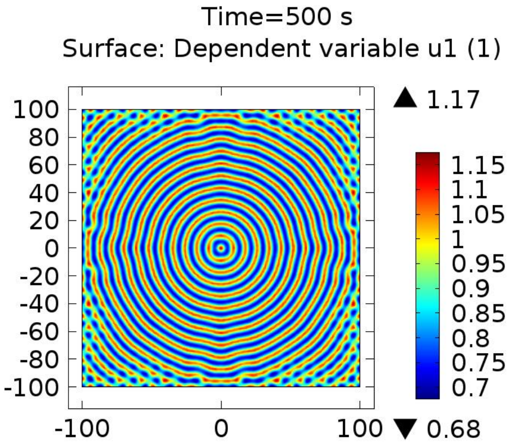


Fig. 4: The numerical solution for Shnakenberg model in (1) using COMSOL which show the pattern formation dynamics in u, when $t = 10$ is a time step and the initial conditions that we use, $u_0(x)$ and $v_0(x)$ are step functions of the form $e^{-x^2-y^2}$ and the parameters are $a_1 = 0.0004$, $a_2 = 0.1$, $c_i = 0.2$, $b = 6.5$, $b_2 = 0.3$, $\alpha = 0.9$ and $\beta = 0.1$

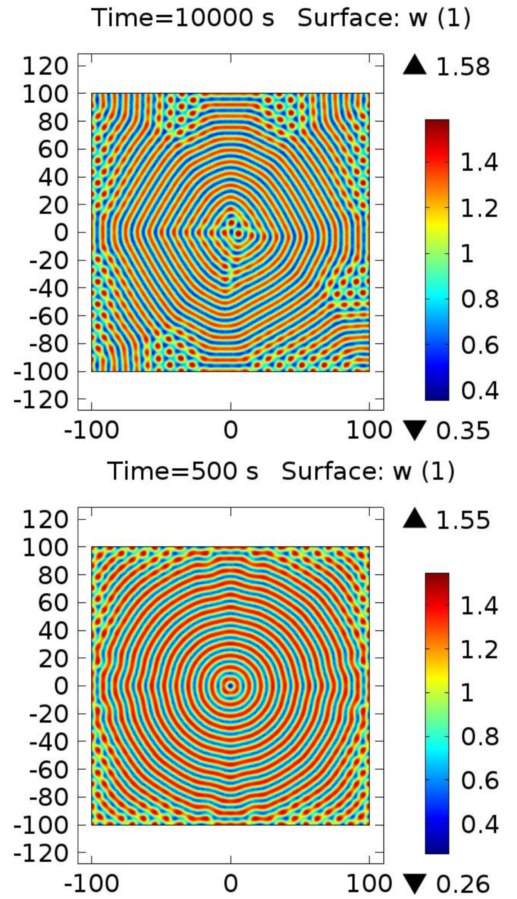


Fig. 5: The numerical solution using COMSOL for Shnakenberg model in (1) which shows the pattern formation dynamics in v, when $t = 10$ is a time step and the initial conditions that we use, $u_0(x)$ and $v_0(x)$ are step functions of the forms $e^{-x^2-y^2}$ and the parameters are $a_1 = 0.0004$, $a_2 = 0.1$, $c_i = 0.2$, $b = 6.5$, $b_2 = 0.3$, $\alpha = 0.9$ and $\beta = 0.1$

5. Conclusion

In this paper, we study the pattern formation for Schnakenberg model with cross diffusion through applying the conditions of diffusion driven instability and we have shown that this model satisfies the four conditions and can formulate the patterns in two dimensions. Also, we found that for specific values of the parameter α , the bifurcation can occur and the limit of the existence of patterns comparing to the wave number values is shown. We used the COMSOL multiphysics software to plot the pattern for these model and shown good results.

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